Determination of altitudes by the trigonometric levelling with different refraction coefficients

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Abstract
Currently, altitude determinations are made using the GNSS technology. It is known that this technology is tributary to the geoid model (quasigeoid). The altitude precision is not always very good for the intended purposes. The geometric levelling is often costly and requires quite a lot of staff. Trigonometric levelling on short distances may be an alternative for determining the altitudes with sufficient precision for most applications. The problem occurs when determining the refraction coefficient, which is usually considered constant. This paper presents a practical way of determining altitudes using the trigonometric levelling method.

Keywords:
trigonometric levelling, refraction coefficient, altitude, precision.

1. Atmospherical refraction

1.1. Overview

Measurement of zenithal or vertical angles can be done with the total station in good conditions. The major problem with these types of measurements is the atmospheric conditions when the point is sighted. In the summer, especially if we are placed in an area with constructions (roads, dams, buildings, etc.), or when the sight passes over a water (river, lake, etc.), the point sighted is seen distorted. The higher the heat, the greater the distortion. The more massive the construction or the larger the water, the greater the distortion.

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For these reasons it is necessary to perform zenithal measurements at certain intervals, taking into account that the Earth's atmospheric refraction cycle is, of course, a direct consequence of the degree of soil heating. The cycle has in principle the following characteristics:

- a maximum, during the night;
- a decrease close to sunrise;
- a minimum, close to 10 o'clock;
- stability between 10h and 15h;
- increase, close to sunset.

As explained above, the value of the refraction coefficient depends also on other factors:

- season;
- geographical position of the area;
- the sighting distance.

For this reason, it is important that the meausrements for zenithal angles (vertical) are made during the period when the refraction coefficient has low values, namely when the temperature is optimal (around 15^0C), in the morning and afternoon. In summer, and especially at noon, it has to be avoided.

1.2. Atmospheric Refraction

Between the state of the soil and the physical properties of the atmosphere, an interaction is created that is reflected in the terrestrial atmospheric refraction.

According to the laws of Snellius and Descartes, in the variant of the terrestrial atmosphere with plane and parallel layers,

- the normal at the separation surface of two transparent homogeneous media, different at the point of incidence, is in the same plane as the incidence ray and the refracted one (Fig. 10.1);
- the ratio between the incidence sinus i to the refraction angle sinus r is a constant magnitude for two given media:

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} = 1 + \frac{\Delta n}{n} \qquad (1)$$

where n_1 and n_2 are absolute refraction indices of the first medium and for the second medium, respectively.

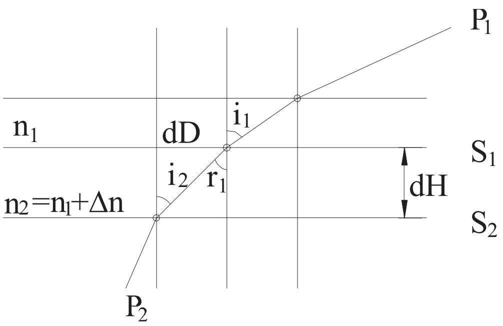

Fig. 1 Refraction of the light rays in the hypothesis of the plane and parallel layers of the atmosphere [Ghiţău - Geodetic Geodesy and Gravimetry – 1983]

We consider that between the points A and B the atmosphere is formed of parallel and very close layers, so it becomes a curved line, visible in Fig.1.2. [1]

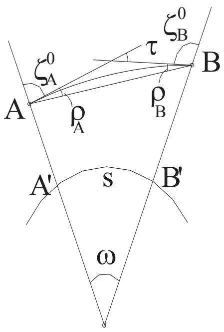

Fig. 2 Vertical terrestrial refraction

AA'O and BB'O represent the direction of the gravity at the two points. From the points A and B the tangents are led to the AB curve line.
Points A' and B' represent the intersection of the gravity direction with the Gauss medium radius sphere.
Thus, the segment AA 'and BB' represent the ellipsoidal altitudes of points A and B.
The magnitudes ζ_A^0 and ζ_B^0 are the angles made by the gravity's directions with the tangents to the curve AB and BA respectively, and they are called zenithal angles or zenithal distances.

The magnitudes ρ_A and ρ_B represent the angle between the direction AB measured in the case there is no refraction coefficient and the tangents to the curve AB and BA respectively, and represents the refraction value in points A and B.
The total refraction is the angle τ of the two tangents and is the sum of the two refraction angles.[1]

$$\tau = \rho_A + \rho_B \tag{2}$$

The correct zenithal angle is given by the relations:

$$\zeta_A = \zeta_A^0 + \rho_A$$
$$\zeta_B = \zeta_B^0 + \rho_B \tag{3}$$

Angle ω represents the angle below which the points A and B of the center of the Earth are visible.

If we mark with Γ the variable curvature of the ray of light, the total refraction can be calculated knowing the relation:

$$\frac{d\tau}{ds} = \Gamma, \tag{4}$$

Finally, we obtain the relation with which is calculated the refraction coefficient:

$$(k-1) = \frac{180^\circ - (\zeta_A^0 + \zeta_B^0)}{\omega}, \tag{5}$$

2. Trigonometric levelling

2.1.Trigonometric geodetic levelling, reciprocal and simultaneous

By reciprocal and simultaneous trigonometric levelling, it is assumed that, at the two ends of the base, respectively in point A and point B, are measured concurrently the two zenithal angles.
Starting from ***Figure 1.2,*** the OAB triangle:

$$\frac{OA}{OB} = \frac{\sin\zeta_B}{\sin\zeta_A} \tag{6}$$

results the formula for determining the level difference between points A and B having measured the zenithal angles ζ_A^0 and ζ_B^0. Also, the heights of the instrument (total station) in points A and B, respectively I_A and I_B, as well as the heights of sight in points A and B, respectively S_A and S_B will be measured. [1]

$$\Delta H_{AB} = S\left(1 + \frac{H_m^0}{R}\right) tg\,\frac{\zeta_B^0 - \zeta_A^0}{2} + \frac{I_A - I_B}{2} + \frac{S_A - S_B}{2} \tag{7}$$

In the formula:

- S is the distance S from the medium radius sphere (ellipsoid);
- $H_m^0$is the mean ellipsoidal altitude between points A and B, roughly calculated from the raw data;
- R - Gauss medium radius.

As can be seen, the influence of the refraction coefficient is null, since having reciprocal sights they mutually cancel.
This type of levelling is difficult to achieve because it involves two operators, two instruments and reciprocal sights.[1]

2.2. Unilateral trigonometric levelling (one sense only)

In the usual works is used the unilateral trigonometric levelling. Even if both a point A and a point B are stationed in a network, they are stationed at different time intervals, so the refraction coefficient is different from one moment of the day to the next. In this case it cannot be considered to be a reciprocal trigonometric levelling.
Thus, it is supposed to station with the instrument at point A and measure the zenithal angle of the station towards station B, respectively ζ_A^0. The formula for calculating the unilateral

trigonometric levelling, starting from the relation (2.2) and considering that:

$$\zeta_B^0 = 180^0 - (\zeta_A^0 - 2\rho - \omega) \quad (8)$$

becomes:

$$\Delta H_{AB} = S\left(1 + \frac{H_A^0}{R}\right) ctg\,\zeta_A^0 + \frac{1-k}{2R\sin^2\zeta_A^0} S^2 + i_A - s_B \quad (9)$$

3. Processing the zenithal observations

3.1. Overview

When zenithal observations are processed, the best method is the least squares method, indirect measurements. In order to determine the most probable values of directly measured magnitudes and unknowns - the indirectly determined magnitudes, the least squares compensation method for indirect measurements is applied.
The unknowns are calculated with functions from the direct measurements. The functions by which the directly determined magnitudes give the unknowns, are of the general form:

$$Mi=f(Xj) \quad (10)$$

For each measurement an equationwill be written. The number of equations must be much higher than the number of the unknowns.
The most probable values are replaced by the measured values. The unknowns with temporarily determined values, values as close as possible to the real value. To these provisional values, the v corrections and the x unknowns respectively are applied.

$$Mi=Mi^0+vi$$
$$Xj=Xj^0+xj \quad (11)$$

in which:

- Mi^0 is the magnitude determined directly, Mi is the most probable value, and vi is the correction;
- Xj^0 is the provisional value of the magnitudes to be determined, Xj is the most probable value, and xj is unknown (correction) applied to the provisional value.

It will be a matrix solution, as follows:
a) the correction equations are written:

$$v=A*x+\ell \quad (12)$$

b)the matrix A, the matrix of coefficients and the matrix of the free terms, l shall be determined. Then the matrices are calculated:

$$N=A^{tr}*P*A \quad (13)$$

where A^{tr} is the transpose of matrix A, P is the weight matrix, and N is the matrix of the system of normal equations, and:

$$L=A^{tr}*P*l \quad (14)$$

c) The unknowns are calculated with the formula:

$$x=N^{-1}*L \quad (15)$$

We also calculate the precisions that tell us how well we measured and how precise are the elevations of the new points.

a') The calculation of the empirical standard deviation of the weight unit s_0 (the mean error of the weight unit):

$$s_0 = \sqrt{\frac{[pvv]}{n-u}} \quad (16)$$

where n is the number of the equations (measurements) and u is the number of the unknowns.

b') The standard deviations of the parameters (unknowns):

$$s_{x_i} = s_0\sqrt{Q_{x_i x_i}} \quad (17)$$

where $Q_{x_i x_i}$ is the weight coefficient extracted from the diagonals of the generalized inverse matrix of the system of normal equations N^{-1}.

c') The standard deviations of the measurements:

$$s_{m_i^0} = \frac{s_0}{\sqrt{p_i}} \quad (18)$$

3.2 Processing the nonreciprocal trigonometric levelling

The direct measurements are the zenithal angles of each station, towards the neighboring points. The indirect magnitudes (unknowns) are the points elevations between each two points among which we have measured the zenithal angle and distance. The refraction coefficient is also considered an unknown.
To obtain the matrix of coefficients, we start from the equation 2.4 and a Taylor series is being developed.. The linearized form of the correction equations is:

$$v_{AB} = a_{AB}dx_A - a_{AB}dx_B + l_{AB} \quad (19)$$

In which:

- v_{AB} is the correction to be applied to the zenithal angle measured from station A towards station B - formula (3.2);
- dx_A is the unknown to be applied to point A – formula (3.2);
- dx_B is the unknown to be applied to point B – formula (3.2);
- l_{AB} is the free term of the equation AB..

If the unknown dk also appears, applied to the refraction coefficient, the equation (18) becomes:

$$v_{AB} = a_{AB}dx_A - a_{AB}dx_B - b_{AB}dk_A + l_{AB} \quad (20)$$

In which:

$$k_A = k_A^0 + dk_A \quad (21)$$

In practice, $k_A^0 = 0{,}13$ for any station.

$$a_{AB} = \frac{1}{t_{AB}}$$
(22)

$$l_{AB}^{0} = \frac{1}{t_{AB}}\left(S_{AB}\cot\zeta_A^0 + \frac{1-k}{2R\,\sin^2\zeta_A^0}S_{AB}^2 - H_B^0 + H_A^0 + I_A - S_B\right) \quad (23)$$

$$t_{AB} = \frac{S_{AB}}{\rho^{cc}\sin^2\zeta_A^0}\left(1+\frac{1-k}{R}S_{AB}\cot\zeta_{AB}^0\right) \quad (3.15)$$

4. Case study

The case study was carried out at the Gura Lotrului hydroelectric power plant. There is a 7-point geodetic network, which was remeasured in autumn 2016, November. Six of the 7 points of the geodetic network were stationed and measurements were made to all visible points. They were measured with 4 series at most points, and with 8 series at the points considered more important. The average of the measurements was calculated as unweighted direct measurements. The precision of the determination of each zenital angle has been calculated with the formulas:

- s_0, the mean square error of a single measurement;;

$$s_0 = \sqrt{\frac{[vv]}{n-1}} \quad (24)$$

and

- s_m, the average square error of the arithmetic mean.

$$s_m = \frac{s_0}{\sqrt{n}} = \sqrt{\frac{[vv]}{n(n-1)}} \quad (25)$$

Where n is the number of the determinations for the zenithal angle.

For each zenithal anglethat was measured, the weight was calculated with the formula:

$$p = \frac{1}{s_m} \quad (26)$$

The stationed points are: S1, S2, S3, S2A, D1 and D2.Point D3 was only sighted from the other points, except S3.

The point which elevation was considered to be fixed was S3 because it had an elevation brought by high precision geometric levelling. The provisional elevations were determined from the provisional level differences.

Tabel 1 Initial data

Station	Point	Mean Zenithal Angle [g c cc]	Weight	Slope Distance [m]
D1	S2	100.31198	0.306144	184.588
	S2A	100.10148	0.186030	209.561
	S3	97.72243	0.152684	269.280
	D3	98.66105	0.165710	194.701
D2	S2	100.30936	0.163382	188.200
	S2A	100.13725	0.120024	157.241
	S3	96.82538	0.136425	193.100
	D3	97.39190	0.200670	99.898
S2	D1	99.78475	0.123899	184.579
	D2	99.78550	0.143592	188.199
	D3	98.66158	0.140633	231.194
	S3	96.65583	0.189903	198.034
	S2A	99.75598	0.162444	112.803
S2A	D1	99.98481	0.135261	209.553
	D2	99.97525	0.076453	157.240
	D3	98.14735	0.148386	147.459
	S3	92.70693	0.142524	85.977
	S2	100.40088	0.124939	112.802
	S1	100.31398	0.123975	169.627
S1	D2	99.75794	0.221649	197.321
	D3	98.79675	0.216085	263.527
	S3	97.37525	0.172989	255.060
	S2A	99.79225	0.123639	169.631
S3	S1	102.69620	0.162221	255.073
	D1	102.34365	0.131024	269.292
	D2	103.26718	0.278468	193.114
	S2A	107.49398	0.119898	86.004
	S2	103.43100	0.112390	198.048

Table 1 presents the mean of the zenithal angles measurements, the weight and the slope distances.

The case study wishes to carry out a study on determining the level differences and in the end the elevations of the new points using the refraction coefficient in three variants:

- A refraction coefficient as a unknown for each station;
- A refraction coefficient as a unknown for the whole network;
- Constant refraction coefficient, 0.13, for each level difference calculated.

Also, to establish values for the refraction coefficient calculated in several variants.

Having the data, the matrix A, described in chapter 3.1, was determined with the coefficients a_{AB} and k_A calculated with the formulas (24-25).

The weight matrix and the free terms matrix have been calculated.

Tabel 2 Difference of the level difference values determined in the 3 ways and the differences between this values

		Level difference [m]			Difference of the level difference [mm]		
Station	Point	A refraction coefficient as a unknown for each station	A refraction coefficient as a unknown for the whole network	No coefficient	1 - 2	1 - 3	2 - 3
D1	S2	-0.7571	-0.7647	-0.7580	-7.6	-1.0	6.7
	S2A	-0.1847	-0.1930	-0.1895	-8.3	-4.8	3.5
	S3	9.7786	9.7711	9.7673	-7.6	-11.3	-3.8
	D3	4.2443	4.2394	4.2405	-4.9	-3.8	1.2
D2	S2	-0.7674	-0.7713	-0.7637	-3.9	3.7	7.6
	S2A	-0.1873	-0.1912	-0.1940	-3.9	-6.7	-2.8
	S3	9.7683	9.7644	9.7616	-3.9	-6.7	-2.8
	D3	4.2340	4.2327	4.2349	-1.3	0.9	2.1
S2	D1	0.7571	0.7647	0.7580	7.6	1.0	-6.7
	D2	0.7674	0.7713	0.7637	3.9	-3.7	-7.6
	D3	5.0014	5.0041	4.9986	2.7	-2.8	-5.5
	S3	10.535	10.535	10.525	0.0	-10.4	-10.4
	S2A	0.5723	0.5717	0.5685	-0.7	-3.9	-3.2
S2A	D1	0.1847	0.1930	0.1895	8.3	4.8	-3.5
	D2	0.1950	0.1996	0.1952	4.6	0.2	-4.4
	D3	4.4290	4.4324	4.4301	3.4	1.1	-2.3
	S3	9.9633	9.9641	9.9569	0.7	-6.5	-7.2
	S2	-0.5723	-0.5717	-0.5685	0.7	3.9	3.2
	S1	-0.7025	-0.6915	-0.6986	11.1	3.9	-7.2
S1	D2	0.8976	0.8911	0.8938	-6.5	-3.7	2.7
	D3	5.1316	5.1238	5.1287	-7.7	-2.8	4.9
	S3	10.6659	10.655	10.6555	-10.4	-10.4	0.0
	S2A	0.7025	0.6915	0.6986	-11.1	-3.9	7.2

S 3	S1	-10.662	-10.659	-10.654	2.6	7.6	4.9
	D1	-9.7786	-9.7711	-9.7673	7.6	11.3	3.7
	D2	-9.7683	-9.7644	-9.7616	3.9	6.7	2.8
	S2A	-9.9633	-9.9641	-9.9569	-0.7	6.5	7.2
	S2	-10.535	-10.535	-10.525	0.0	10.4	10.4

For each of the tree cases presented above there is a variant of matrix A, which has the following dimensions:
28 equations with 12 unknowns. The 28 equations are the 28 zenithal angles measured in the 6 stations. The 12 unknowns are: 6 elevations of the new points and 6 refraction coefficients, one for each point stationed.
28 equations with 7 unknowns. The 28 equations are the 28 zenithal angles measured in the 6 stations. The 7 unknowns are: 6 elevations of the new points and 1 refraction coefficient for the whole network.
28 equations with 6 unknowns. The 28 equations are the 28 zenithal angles measured in the 6 stations. The 6 unknowns are the 6 elevations of the new points. The refraction coefficient is considered constant.
The weight matrix is the same in all 3 variants and has the dimensions 28/28. The free terms matrix is the same in all 3 variants and has 28 lines and a column.

4. Conclusions

The case study was carried out on a network that is not very large, the longest distance being of 256m. For this reason, the results are quite close in the 3 variants. The differences between the 3 variants are of a maximum of 11mm. We have to take into account that the heights of the instruments and the sight elevations were not measured with a precision better than 2 - 3mm.
The instrument used was the TS 06 Plus, Leica total station, which ensure the minimum reading of1^{cc}.
The refraction coefficients is the one having very large variations from the reference value of 0.13. Thus, the highest value of the coefficient is of 5.81, and the lowest of -0.23. This leads to the conclusion that the refraction coefficient can be different from hour to hour, function to the weather conditions. Thus, measurements were made over three days, the temperatures were very different, from 0^0 up to $+15^0$C. Also, there were moments when the humidity was very high, and moments when the humidity lowered.

References

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