Deformation prediction and analysis by finite element method

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Abstract
This article presents the basic idea in the finite element method, which is to find a solution of a complicated problem by breakdown to several simpler problems. The main issue is that since the actual problem is replaced by a simpler one, it will be able to find an approximated solution rather than the exact one, the approximation is strong related to the type of the discretization and the number of finite elements use to solve the problem.
Although this method has been extensively used in other engineering fields like structural mechanics, fluid dynamics, heat conduction and electrical field it can be successfully applied to solve several problems in deformation analysis.

Keywords
Finite element method, Prediction, Deformation Analysis

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1. Introduction

Deformation analysis involves studying the displacement of network points in several measurement stages. Thus, there are surveyed sizes as directions, distances, level differences, the respective networks are compensated by means of some methods in accordance with the requirements, then the next step is the deformation analysis using the global congruence test and it is determined by the application of some localization statistical tests. During these tests, the limit values are determined with the help of some tables depending on certain parameters and values for each point. The two values are then compared and the result obtained is used to determine whether a point is displaced or not. In our days it surveying blocks can be analysed performed in several stages, while at first just two models were compared. [1].
If we have a geodetic network that was carried out in order to obtain the coordinates of a set of geodetic points and then used as a reference point for subsequent topographical works, the measurements are carried out and processed only once. In a geodetic network carried out to determine the displacement or the deformation of lands or buildings, the measurements are carried out and processed in several stages between which there are determined the displacements as differences in coordinates. In reference, it's proposed the block processing of the measurements carried out in several stages in a tracking network. [2]
The basic principle in the finite element method (MEF) is finding a solution for a complex problem by replacing it with a simpler one. By this replacement it can be found just an approximate solution of the problem instead of the exact solution. This method can be applied in the cases in which no mathematical model can't solve the problem by finding a precise solutions, but there are cases when there can't be found nor approximate solutions by other methods. Thus, in the absence of any other ways to bring about a resolution of the problem, even if it's approximate, the finite element method is a method which shall be taken into account. It

has, on the other hand, the advantage that the solution can be improved by applying a bigger volume of calculation by dividing the problem in a larger number of finite elements in order to obtain an approximate solution.

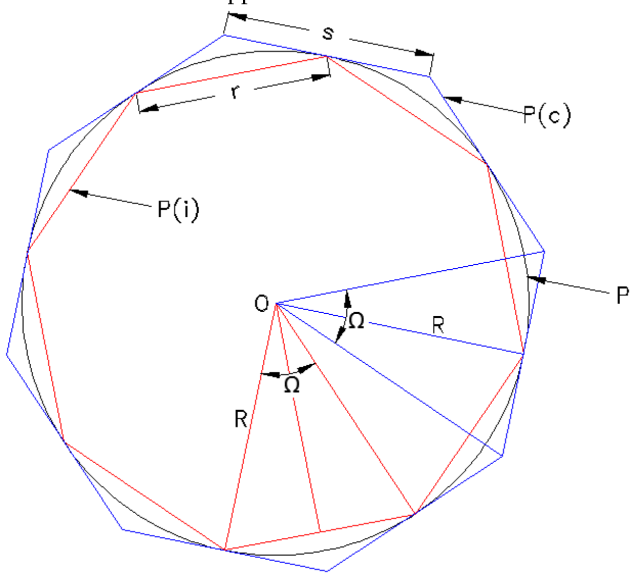

Fig. 1 The circle perimeter approximation using MEF

In the figure above the letterings represent:
- *P*, *P(i)*, *P(c)* – the perimeter of the circle, the perimeter of the polygon inscribed in the circle, respectively the perimeter of the circumscribed polygon;
- *r*, *s* – length of side of the polygon inscribed, respectively circumscribed in the circle.

When using the finite element method, the final solution for the problem of the studied object is considered to be the sum of the individual solutions of each element. A simple example for the way in which the finite element method can be used is the approximation of the perimeter of a circle with the perimeter of a polygon with *n* number of sides. The polygon can be circumscribed as well as inscribed in the circle, as it can be seen in fig. 1.

In order to demonstrate that the perimeter of a circle is approximated using the perimeter of a polygon with *n* number of sides, it must be demonstrated that:

$$\lim_{n \to \infty} P(i) = P \tag{1}$$

$$\lim_{n \to \infty} P(c) = P \tag{2}$$

For this, the perimeter of the two polygons must be expressed in accordance with the radius of the circle, *R*, and the number of sides, *n*. Thus, for each side of the polygons it can be used the following relations:

$$r = 2R\sin\frac{\pi}{n} \tag{3}$$

$$s = 2R\tan\frac{\pi}{n} \tag{4}$$

Taking into consideration that the perimeter of a polygon is the product between the length of side with the total number of sides, it can be said that:

$$P(i) = nr = 2nR\sin\frac{\pi}{n} \tag{5}$$

$$P(c) = ns = 2sR\tan\frac{\pi}{n} \tag{6}$$

The above relations can be transcribed in this way:

$$P(i) = 2\pi R\left[\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right] \tag{7}$$

$$P(c) = 2\pi R\left[\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\right] \tag{8}$$

As *n* tends to infinity, it follows that π/n tends to *0*, so:

$$P(i) = 2\pi R = P \tag{9}$$

$$P(c) = 2\pi R = P \tag{10}$$

This demonstration shows that, despite the fact that the finite element method is a method that has an approximate result, the growth of the number of finite elements is directly proportional with the precision of the method. It can be said that when the number of finite elements tends to infinity, the method doesn't have an approximate solution, but an exact one.

2. Historical Background

Even if the name of the finite element method has been recently assigned, the concept has dated for a few centuries. For example, the calculation of the circle circumference through approximation with the perimeter of a polygon was realized by mathematicians of the Ancient times. Thus, in nowadays terms, each side of the polygon is a finite element, and as the number of the finite elements increases, the approximate value converges to the real value. These characteristics are generally valid in every application of the finite elements. [6]

In order to find the differential equation of a minimum surface area delimited by given closed curve, Schellback discretized the area in many triangles and he used an expression of a finite difference in order to calculate the total discretized area in 1851. In the finite element method, a differential equation is solved replacing it with a set of algebraic equations.

Since the 1900s, the behaviour of structural frames, composed of many bars arranged in a regular way, was approximated through the behaviour of an elastic isotropic body.
In 1943, Courant presented a method to determine the rigidity at torsion for a tubular tree through the graduation of the transversal section in many triangles and by using a linear variation of the torsion function for each triangle in its characteristic points (called node in the terminology of this method). This study is considered to be the origin of the finite elements method.
Since the 1950s, the aeronautics industry engineers have begun to work at the development of an approximate method of prediction for the forces exerted by the wind on aircraft wings.
The name *finite elements* was used, for the first time, by Ray William Clough, considered to be the founder of the method. Even if the method was initially developed through intuition and mathematical arguments, it was recognised to be a form of the classical method Rayleigh-Ritz at the beginning of the 1960s. As the same time as the recognition of the mathematical base of the method, the development of new methods of the finite elements for different kind of problems increased the popularity of the method.
The fact that both had a fulminant and IT development led to the validation of the method in a practical and can be achieved high volume calculation method that is part of a more or less automatically.
A broader interpretation of the finite element method was carried out by Zienkiewicz and Cheung and applicability of the method in any field. With this interpretation has been calculated that extended finite element method can be derived by applying a weighted residual methods such as Galerkin method or the method of minimum squares. This expanded interest mathematicians and engineers in applying the finite element method for finding the solution of various linear or non-linear differential equations.
Traditionally, mathematicians are those who develop techniques and methods for finding solutions of differential equations and engineers use these methods to solve problems in various fields. Only if the finite element method are the engineers who developed the technique for mathematicians to use the method for finding the solution of complex differential equations. Nowadays, solve engineering problems by finite element method has become an industry, using the method allowed millions of degrees of freedom.
The rapid progress of the finite element method can be seen by the fact that annually about 3,800 research papers are presented and published. Since 1995 were estimated a total of 56,000 publications, 380 books and 400 conferences in the finite element method.
By this progress today finite element method is considered as the most well-established and convenient method of analysis by both engineers and the scientists.

3. Presentation of the method

The finite element method is a method of processing approximate differential equations partial differential equations that describe physical phenomena or not real.
In principle, finite element analysis is to divide the area into zones of simple geometric form, their analysis and recomposition domain under certain mathematical goals.
Areas of application of the finite element method can be any fields that describe a phenomenon using differential equations. Until now, the method has developed remarkably in areas such as fluid analysis, analysis of electrical, magnetic analysis, thermal analysis, and structural analysis. [6]
Calculation program used to analyse the problem does not solve the actual structure, but its models. Modeling is an activity to simplify the structure by trying to fit various portions of simplifying structures and bearing loads. Modeling is directly related to the model and is very much everyone's experience, inspiration and knowledge is closely related to the theoretical basis of the method. Every program that is based on the finite element method has some peculiarities that must be learned but there are some basic rules that once mastered the method allows addressing any program that is based on the finite element method.

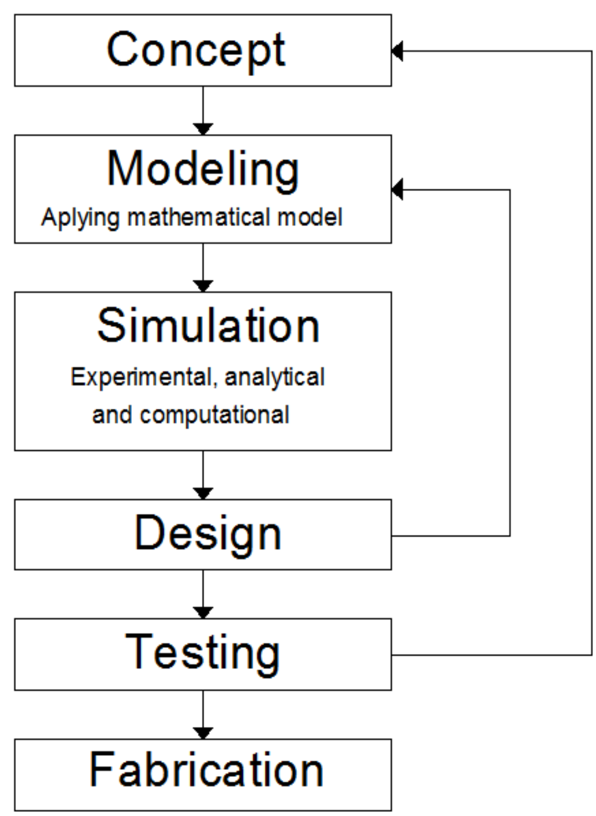

Fig. 2 The processes performed before execution

The finite element method has developed into an indispensable technology in modeling and simulation of advanced engineering systems. In an attempt to build an advanced engineering, engineers and designers go through a sophisticated process which consists of modeling, simulation, visualization, analysis, design, testing and finally production. Most often a huge amount of work is done before the actual manufacture of construction. This is done to ensure the optimum use of the construction, but also to have a low cost of the construction project. This process is illustrated in Figure 2, the process that is taken up several times in different phases due to the data obtained in the field. The techniques that are directly related to modeling and simulation plays an important role, and finite element method is a standard tool used in almost any large scale project.

The finite element method was initially used to solve problems of structural analysis and analysis of solids and has since been used to solve different types of problems in various branches of engineering. In principle, the finite element method is a numerical method that seeks an approximate solution distribution in the investigated variables, which is usually difficult to obtain analytical methods. Apply by dividing the domain into a finite number of pieces called elements usually simple shapes in geometrical point of view.

Figure 3 illustrates schematically the distribution of a function f (x), where it is approximated using unidimensional finite element method. In this case f (x) is a continuous function is approximated using piecewise linear functions in each item. If unidimensional end of each element is called node. Unknown variables in finite element discrete function values are in knots. Mathematical Principles are so used to establish equations for each element, then the elements are related to each other to describe the distribution of the whole geometry. [6]

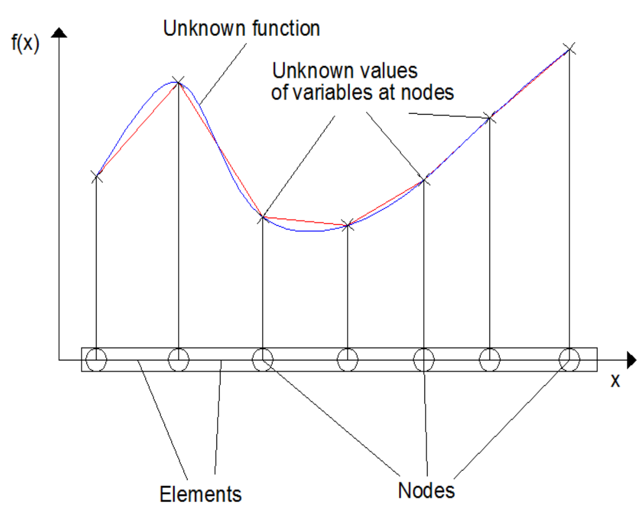

Fig. 3 The finite element method approximation

Steps to solve a problem using the finite element method.

1. Dividing the domain of finite element analysis

At this stage, choose the type of finite elements according to the subject studied and divide the structure into finite elements. This operation is called meshing and often is done automatically by computer. Finite element type is defined by several aspects such as the number of dimensions (one-dimensional, two-dimensional or three-dimensional); number of nodes of each element; Approximation functions to be associated. The quality of results is directly influenced by the choice of finite elements and the way it is meshed object studied.

2. Establishment of finite element equations

The behaviour of the studied object in the body of a finite element equations described by finite elements, also called elemental equations. They form a system of equations for each finite element.

3. *Assembling the elemental equations in the system of equations of structure*

The behaviour of the whole structure is modelled by assembling systems of equations for each finite element in the overall system of equations of the structure, which means that the balance is directly proportional to the equilibrium structure of finite elements. By joining their common nodes requires that the unknown functions have the same value.

4. *Implementation of boundary conditions and solving the structure system of equations*

The system of equations thus formed is solved to obtain the values of the functions in each node of the network. These are called primary or first-order unknown.

5. *Perform additional calculations to determine the secondary unknown*

In some cases, after determining the primary unknowns, the analysis ends. This is usually the case for problems where the unknowns are to be found in the nodes formed by meshing. There are cases, however, the primary unknowns just knowing is not enough, the analysis continues to finding some unknowns secondary or second order. Thus, for example, in monitoring deformations of primary unknowns are the displacements of nodes. With their side is determined the secondary unknowns that are specific deformations in every node.

4. Mathematical model

A mathematical model for the analysis of a structure involves automatically determining a number of variables and functions $u^*(\xi)$ representing the displacements, strains, ξ coordinate the functions ξ (x, y, z), defined nodal points. If the exact solution $u^*(\xi)$ is unknown and u (ξ) is an approximation of it, where the error function e (ξ) is:

$$e(\xi) = u(\xi) - u^*(\xi) \quad (11)$$

To find an approximate solution of the problem is sufficiently to describe an expression which contain *n* parameters of approximation.

$$u(\xi) = u(\xi, a_1, a_2, ... a_n) \quad (12)$$

This will determine these parameters based on the error function.

The finite element approximation is nodal and has the following form:

$$u(\xi) = [N_1(\xi) N_2(\xi) N_3(\xi) ... N_N(\xi)] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} = [N]^T (u_n) \quad (13)$$

Where:

- u_i are nodal parameters of approximation and has a concrete physical significations (nodal displacement),
- Ni(ξ) – interpolation or approximation functions, which usually has polynomial forms.

The approximate solutions *u(ξ)* can be built on the entire structure domain of definition *V* or on elementary subdomains V_e of the finite elements which means that:

$$V = \sum_{e=1}^{n} V_e \quad (14)$$

Each finite element is numbered, usually from 1 to the total number of elements *n*, reference to an element is the index *e*., The operation of a field in a finite number of nodes and finite elements, whether the finite elements are alike or not, is called meshing.

Meshing domain analysis is the first step to be performed in numerical solving a problem using the finite element method. Meshing involves deciding principle forms the finite element size and type, but all these are strictly related to the element studied.

Each type of finite element should be designed to satisfy the most closely studied the mesh object and there is not a finite element "generally valid", with which you can model any geometric shape. [3]

5. Meshing domain analysis

Engineers are often faced with the delicate problem of the relationship between the numerical solution quality and computational effort made to obtain the solution. In general, the mesh size is directly proportional to the accuracy of the solution. However a large network of finite elements leads to large systems of equations, whose solution is often difficult. This is particularly important in determining the number of finite elements mesh mode domain analysis the analyst has to be, by experience, to decide on increasing the mesh only in areas where large gradients are expected of unknowns. In areas where such variation is considered small to insignificant impact can be achieved minimal networks of finite elements so that the result is about the same accuracy. [6]

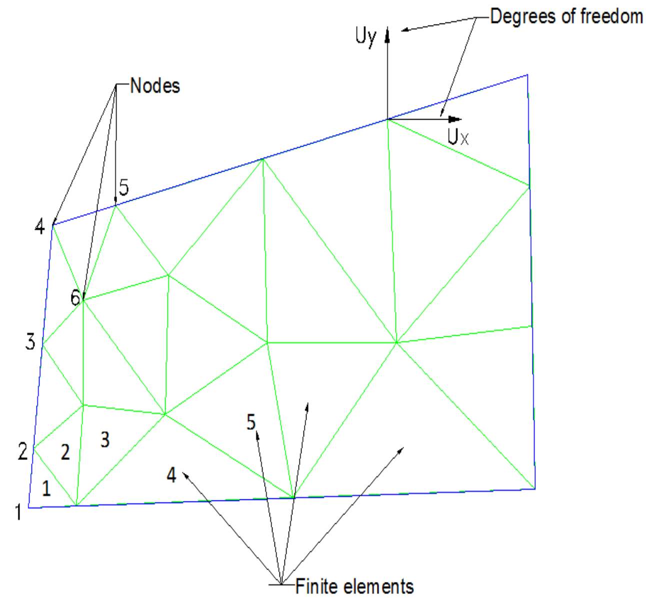

Fig. 4 Analysis domain discretization

Each node of the mesh thus formed shows a possible displacement on ox axis and on oy axis, so one can say that there are two parameters that define a uniquely moving a node in two-dimensional space. These parameters are called degrees of freedom of a node, in literature degrees of freedom are denoted by *DOF*.

The calculation model of a structure to be subjected to an analysis by finite element method, where generally consists of lines, flat or curved or volumes. At this stage of modeling, the model is continuous with an infinite number of points. Meshing is the fundamental approach in the finite element method and consists of converting a continuous structure with an infinite number of points, a discrete model, model characterized by a finite number of points, called nodes.

The nodes are defined primary nodal unknowns whose values are calculated by finite element method. Uncertainties associated nodes can be trips, where the finite element model is called displacement. To shift the pattern is recognized that the deformed shape of a structure, as a result of the various requirements, is defined by the displacements of all nodes in relation to the network node prior to deformation. Each node can have up to six components of the movement, this being called nodal displacements, with respect to a reference coordinate system and the system to which it is reported the overall structure of the three components x, y, z, and three rotations of the linear movement Φx, Φy, Φz.

6. Types of finite elements

A general classification, rather vague classification, finite elements divided into three categories, depending on the number of dimensions which constitute the mesh in which it is used:
- Unidimensional finite elements;
- Two-dimensional finite elements;
- Three-dimensional finite elements.

Unidimensional finite elements are used when the size considered to be approximated depend on a single variable, so that the items are one-dimensional line segments or arcs curve along which takes values independent variable of the problem.

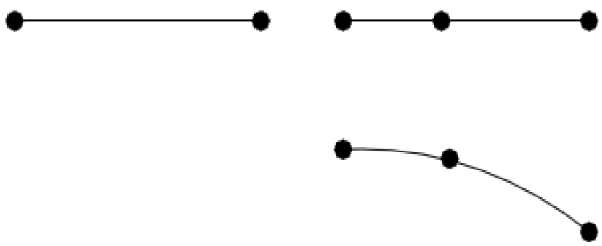

Fig. 5 Unidimensional finite elements

Dimensional finite elements are used when the approximate size directly depends on two variables. For example, they can be used to resolve the two-dimensional elasticity, deformation and plane displacement.

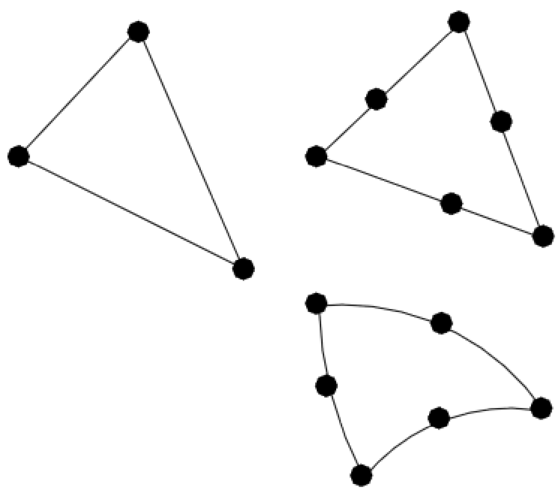

Fig. 6 Two-dimensional finite elements

Three-dimensional finite elements form a fairly large class, from this class increasingly common are the following two families of finite elements: tetrahedral finite elements and hexahedral finite elements.

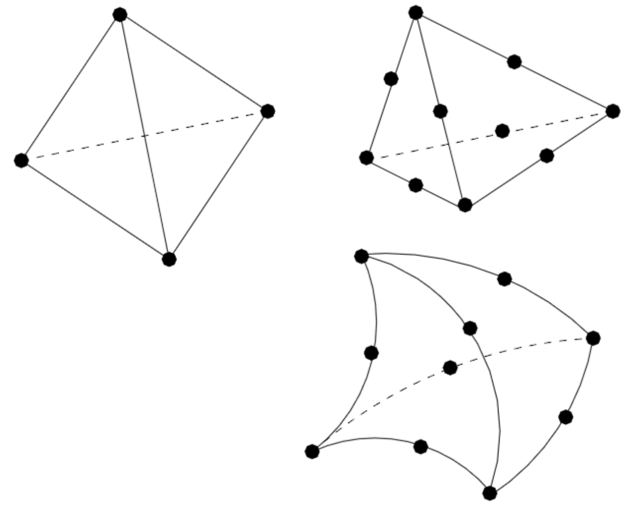

Fig. 7 Three-dimensional finite elements

In practice are rarely used elements function approximation of degree greater than three. The great diversity of elements currently in use leads to inability to find a single criterion for classification. In particular, high mathematical complexity finite elements are the most difficult to include in a certain class of items.

7. Advantages and disadvantages

The main advantages of the finite element method are:
- Flexibility – this feature is given by the fact that it allows meshing of bodies or surfaces no matter how complex these are in elements of geometric shapes not only simple, but also very popular, and by applying basic math calculations can calculate local variables for each finite element separately which will be use them in determining global variables and thus solve the problem and solution.
- The possibility of modelling the inhomogeneous bodies considering their physical properties
- Ease to implement finite element method in general computer programs.
- The possibility of applying the same method for different types of problems in various engineering fields due to the generalization of the method.
- The simplicity of the basic concepts of the finite element method can be considered an important advantage of the method. The importance of ownership and understanding them correctly may result from the fact that such concepts include certain assumptions, simplifications and generalizations. But ignoring these concepts can lead to serious errors in modeling and finite element analysis and thus can become a disadvantage of the method.

The finite element method disadvantages:
- A major disadvantage of the finite element method can be considered the large number of input data, the large amount of computation to be performed to calculate the

model. The largest share of the workload is meshing the object into finite elements and calculating the coordinates of nodes and the configuration of finite elements.

Currently processing and analysis programs by finite elements method on the market have specialized modules for automatic discretization, thus relieving the user from making a large amount of work by manual meshing.

8. Conclusions

The finite element method is a very popular analysis method and it is used in several engineering domains. This method presents the advantages that the studied object can be discretized in simple finite elements, no matter how complex the object is, the possibility to apply this method in several domains and the simplicity of the base concepts of the method. On the other hand, the breakdown of the object in an extremely large number of finite elements, to get closer to the true result, lead to the formation of an equally large number of nodes and hence an extraordinary volume of calculation.

If the computer technology didn't evolve at the same time with the method, facilitating the huge work, the lack of precision of determination would made the method useless for many branches of engineering, including engineering geodesy measurements.

The objects discretization in finite elements is the part where the largest volume of work is filled, this part is easier with specialized programs that have implemented software which automatically performs mesh, regardless of the complexity of the studied objects.

The advantages of the finite element method, described above , provides the opportunity to study this method not only in predicting deformations or displacements but also in processing and analyzing measurements.

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